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FRACTURE OF MATERIALS UNDER COMPRESSION ALONG A PERIODIC SYSTEM OF CRACKS UNDER PLANE STRAIN CONDITIONS*

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Elastic deformations in materials are considered within the framework of a three-dimensional linearized stability theory. On the basis of a fracture criterion proposed earlier /1, 2/ critical values are specified for the compression strain and stress under plane strain corresponding to the beginning of material fracture under compression along a periodic system of cracks.

1. We consider an infinite series of cracks

$$\{x_2 = (2hn),\ |x_1| < a, -\infty < x_3 < +\infty\}, \quad n = 0, \pm 1, \pm 2, \ldots$$

(x_j are Lagrange coordinates that agree with Cartesian coordinates in the unstrained state). Because of compression in the direction of the x_1 axis (parallel to the planes of the cracks), a homogeneous subcritical state occurs in the material /2/

$$\begin{gathered}\sigma_{ii}^{\circ} = \text{const}, \quad \sigma_{22}^{\circ} = 0, \quad \sigma_{11}^{\circ} \neq 0, \quad \sigma_{33}^{\circ} \neq 0 \\ u_m^{\circ} = \delta_{im}\ (\lambda_i - 1)\ x_i, \ \lambda_i = \text{const}, \ \lambda_3 = 1 \ (i = 1, 2, 3)\end{gathered} \tag{1.1}$$

(λ_i are elongations along the axes, $\lambda_1 < 1$, δ_{ij} is the Kronecker delta, and $\boldsymbol{\sigma}^{\circ}$ is the symmetric stress tensor). The perturbation of the displacement vector $\mathbf{u}$ and the non-symmetric Kirchhoff stress tensor $\mathbf{t}$ are used in formulating the linearized problem.

The crack edges are stress-free. The boundary conditions for the linearized problem are written in the form (the crack edges are marked by the plus and minus subscripts)

$$t_{22} = 0, \ t_{21} = 0 \ \ (x_2 = (2hn)_{\pm}, \ |x_1| < a, \ \ n = 0, \pm 1, \pm 2, \ldots) \tag{1.2}$$

In connection with the periodicity of the geometric and force diagrams of the problem, it is sufficient to examine just the layer $|x_2| \leqslant h$. Because of the symmetry of the configuration about the plane $x_2 = 0$, we represent the stress and displacement fields in the form of sums of a symmetric part (symmetric buckling mode) and an antisymmetric (bending buckling mode) part relative to this plane, and because of the linearity of the problem we will investigate them separately. Consequently, the problem is reformulated for the layer $0 \leqslant x_2 \leqslant h$ as follows:

$$\begin{gathered}u_{\alpha} = 0 \ (x_2 = 0, \ |x_1| > a), \ t_{2\alpha} = 0 \ (x_2 = 0, \ |x_1| < a) \\ t_{2\gamma} = 0 \ (x_2 = 0, \ |x_1| \geqslant 0), \ u_{\alpha} = 0, \ t_{2\gamma} = 0 \ (x_2 = h, \ |x_1| \geqslant 0)\end{gathered} \tag{1.3}$$

where $\alpha = 2, \gamma = 1$ for the symmetric buckling mode, and $\alpha = 1, \ \gamma = 2$ for the bending mode.

2. We will carry out the investigation in general form for the theory of large, and two versions of the theory of small, subcritical strains, compressible and incompressible bodies with an arbitrary kind of elastic potential for the cases of equal and unequal roots of the characteristic equation (the terminology of /2/).

We select the general solutions of the equations of the linearized theory for the subcritical state (1.1) in the following form /2-4/.

For equal roots ($n_1 = n_2$)

$$\begin{aligned}u_1 &= -\frac{\partial\varphi}{\partial x_1} - z_1\frac{\partial F}{\partial x_1}, \\ u_2 &= n_1^{-1/2}\left[(m_1 + 1 - m_2)F - m_1\Phi - m_1 z_1\frac{\partial F}{\partial z_1}\right] \\ t_{21} &= c_{44}n_1^{-1/2}\frac{\partial}{\partial x_1}\left[(d_1 - d_2)F - d_1\Phi - d_1 z_1\frac{\partial F}{\partial z_1}\right] \\ t_{22} &= c_{44}\left[(d_1 l_1 - d_2 l_2)\frac{\partial F}{\partial z_1} - d_1 l_1\frac{\partial \Phi}{\partial z_1} - d_1 l_1 z_1\frac{\partial^2 F}{\partial z_1^2}\right]\end{aligned} \tag{2.1}$$

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$$z_1 = n_1^{-1/2}x_2,\ \Phi = \partial\varphi/\partial z_1;\ d_1 = \omega_{2112}\omega_{2121}^{-1} + m_1$$
$$d_2 = 2\omega_{2112}\omega_{2121}^{-1} + m_2 - 1,\ c_{44} = \omega_{1212}$$

Here the quantities $c_{44}, n_i, m_i, l_i, d_i$ belong to the plane problem in contrast to those utilized earlier /2, 3/ for the spatial problems.

The potentials F, φ, Φ satisfy the Laplace equation in the variables x_1, z_1. The quantities n_1, m_i, l_i are determined for compressible bodies /2, 4/ by the equalities

$$\begin{aligned}
n_1 &= (\omega_{2222}\omega_{2112}\omega_{1111}^{-1}\omega_{1221}^{-1})^{1/2}\\
m_1 &= (\omega_{1111}n_1 - \omega_{2112})(\omega_{1122} + \omega_{1212})^{-1}\\
m_2 &= (\omega_{1122} + \omega_{1212} - 2\omega_{2112})(\omega_{1122} + \omega_{1212})^{-1}\\
l_1 &= (-n_1\omega_{2211} + m_1\omega_{2222})\, n_1^{-1}(\omega_{2112}\omega_{2121}^{-1} + m_1)^{-1}\omega_{1212}^{-1}\\
l_2 &= [-n_1\omega_{2211} + (m_1 + m_2 - 1)\,\omega_{2222}]n_1^{-1}(2\omega_{2112}\omega_{2121}^{-1} +\\
&\quad m_2 - 1)^{-1}\,\omega_{1212}^{-1}
\end{aligned} \tag{2.2}$$

For incompressible bodies

$$\begin{aligned}
n_1 &= q_{22}q_{11}^{-1}(\varkappa_{2112}\varkappa_{1221}^{-1})^{1/2},\quad m_1 = q_{11}q_{22}^{-1}n_1,\ m_2 = 1\\
l_1 &= [m_1\varkappa_{2222} + n_1(q_{11}^{-1}q_{22}\varkappa_{1111} - \varkappa_{1212} - 2\varkappa_{1122}) -\\
&\quad q_{11}^{-1}q_{22}\varkappa_{2112}]\, n_1^{-1}(\varkappa_{2112}\varkappa_{2121}^{-1} + m_1)^{-1}\varkappa_{1212}^{-1}\\
l_2 &= [m_1\varkappa_{2222} + n_1(q_{11}^{-1}q_{22}\varkappa_{1111} - \varkappa_{1212} - 2\varkappa_{1122}) -\\
&\quad 3q_{11}^{-1}q_{22}\varkappa_{2112}]n_1^{-1}(2\varkappa_{2112}\varkappa_{1212}^{-1} + m_2 - 1)^{-1}\,\varkappa_{1212}^{-1}
\end{aligned} \tag{2.3}$$

with the component of the tensor ω in (2.1) replaced by corresponding components of the tensor $\varkappa$.

For unequal roots $(n_1 \neq n_2)$

$$\begin{aligned}
u_1 &= \frac{\partial}{\partial x_1}(\varphi_1 + \varphi_2),\quad u_2 = m_1 n_1^{-1/2}\frac{\partial\varphi_1}{\partial z_1} + m_2 n_2^{-1/2}\frac{\partial\varphi_2}{\partial z_2}\\
t_{21} &= c_{44}\frac{\partial}{\partial x_1}\left[d_1 n_1^{-1/2}\frac{\partial\varphi_1}{\partial z_1} + d_2 n_2^{-1/2}\frac{\partial\varphi_2}{\partial z_2}\right]\\
t_{22} &= c_{44}\left(d_1 l_1\frac{\partial^2\varphi_1}{\partial z_1^2} + d_2 l_2\frac{\partial^2\varphi_2}{\partial z_2^2}\right)\\
z_i &= n_i^{-1/2}x_2,\quad d_i = \omega_{2112}\omega_{2121}^{-1} + m_i,\quad i = 1,\ 2;\ c_{44} = \omega_{1212}
\end{aligned} \tag{2.4}$$

The potentials φ_i satisfy the Laplace equation in the variables x_1, z_i, $i = 1, 2$.
The quantities n_i, m_i, l_i are determined for compressible bodies /2, 4/ by the equalities

$$\begin{aligned}
n_{1,2} &= c \pm (c^2 - \omega_{2112}\omega_{2222}\omega_{1221}^{-1}\omega_{1111}^{-1})^{1/2}\\
2c\omega_{1111}\omega_{1221} &= \omega_{1221}\omega_{2112} + \omega_{1111}\omega_{2222} - (\omega_{1122} + \omega_{1212})^2\\
m_i &= (n_i\omega_{1111} - \omega_{2112})(\omega_{1122} + \omega_{1212})^{-1},\ l_i = (m_i\omega_{2222} -\\
&\quad n_i\omega_{2211})(\omega_{2112}\omega_{2121}^{-1} + m_i)^{-1}n_i^{-1}\omega_{1212}^{-1},\ i = 1,\ 2
\end{aligned} \tag{2.5}$$

For incompressible bodies

$$\begin{aligned}
n_{1,2} &= c \pm (c^2 - q_{22}^2q_{11}^{-2}\varkappa_{2112}\varkappa_{1221}^{-1})^{1/2}\\
2c\varkappa_{1221} &= \varkappa_{2222} + q_{11}^{-2}q_{22}^2\varkappa_{1111} - 2q_{11}^{-1}q_{22}(\varkappa_{1122} + \varkappa_{1212})\\
m_i &= q_{11}q_{22}^{-1}n_i,\ l_i = [m_i\varkappa_{2222} + n_i(q_{11}^{-1}q_{22}\varkappa_{1111} - 2\varkappa_{1122} -\\
&\quad \varkappa_{1212}) - q_{11}^{-1}q_{22}\varkappa_{2112}](\varkappa_{2112}\varkappa_{2121}^{-1} + m_i)^{-1}n_i^{-1}\varkappa_{1212}^{-1},\\
i &= 1,\ 2
\end{aligned} \tag{2.6}$$

with the components of the tensor ω in (2.4) replaced by the components of the tensor $\varkappa$.

The determination of the tensors ω for compressible bodies and $\varkappa$ (as well as the quantities q_{ij}) for incompressible bodies is discussed in detail in /2/, appropriate simplifications are introduced in the procedure for their determination here when examining the two versions of the theory of small subcritical strains.

3. Taking into account the symmetry of the displacement field about the x_2 axis, we represent the potential in the representation of the general solutions (2.1) and (2.4) in the form of Fourier integral cosine expansions in the coordinate x_1 and consider the values $x_1 \geqslant 0$.

For equal roots

$$\varphi(x_1, z_1) = -\int_0^\infty [B_1(\lambda)\,\mathrm{sh}\,\lambda(h_1 - z_1) + B_2(\lambda)\,\mathrm{ch}\,\lambda(h_1 - z_1)]\,(\lambda\,\mathrm{sh}\,\lambda h_1)^{-1}\cos\lambda x_1\, d\lambda \tag{3.1}$$

$$F(x_1, z_1) = \int_0^\infty [A_1(\lambda)\operatorname{ch}\lambda(h_1 - z_1) + A_2(\lambda)\operatorname{sh}\lambda(h_1 - z_1)](\operatorname{sh}\lambda h_1)^{-1}\cos\lambda x_1\, d\lambda$$

which corresponds to evenness of the displacement u_2 in x_1 and unevenness of u_1.

For unequal roots

$$\varphi_1(x_1, z_1) = \int_0^\infty [A_1(\lambda)\operatorname{ch}\lambda(h_1 - z_1) + A_2(\lambda)\operatorname{sh}\lambda(h_1 - z_1)](\lambda\operatorname{sh}\lambda h_1)^{-1}\cos\lambda x_1\, d\lambda \tag{3.2}$$

$$\varphi_2(x_1, z_2) = \int_0^\infty [B_1(\lambda)\operatorname{ch}\lambda(h_2 - z_2) + B_2(\lambda)\operatorname{sh}\lambda(h_2 - z_2)](\lambda\operatorname{sh}\lambda h_2)^{-1}\cos\lambda x_1\, d\lambda$$

$$(h_i = hn_i^{-1/2}, \quad i = 1, 2)$$

The boundary conditions given in the whole $x_2 = \mathrm{const}$ plane (the last three conditions in (1.3)) afford the possibility of preserving just one out of the four unknown functions $A_i(\lambda)$, $B_i(\lambda)$ $(i = 1, 2)$ in the integral expansions (3.1) and (3.2). The remaining two boundary conditions enable the problem to be reduced to systems of two pairwise integral equations for the symmetric and bending buckling modes, respectively.

The bending buckling mode. Taking (2.1) and (2.4) into account, the first two boundary conditions in (1.3) follow from the relationships:

For equal roots

$$\varphi = 0 \ (z_1 = 0,\ a < x_1 < \infty) \tag{3.3}$$

$$(d_1 - d_2)F - d_1\Phi = 0 \ (z_1 = 0,\ 0 \leqslant x_1 < a)$$

For unequal roots

$$\varphi_1 + \varphi_2 = 0 \ (z_i = 0,\ i = 1, 2;\ a < x_1 < \infty) \tag{3.4}$$

$$d_1 n_1^{-1/2}\frac{\partial\varphi_1}{\partial z_1} + d_2 n_2^{-1/2}\frac{\partial\varphi_2}{\partial z_2} = 0 \quad (z_i = 0, i = 1, 2; 0 \leqslant x_1 < a)$$

On the basis of (3.3) and (3.4) we obtain systems of dual integral equations in the form

$$\int_0^\infty \lambda A(\lambda)[1 - g(\lambda)]\cos\lambda x_1\, d\lambda = 0 \quad (0 \leqslant x_1 < a) \tag{3.5}$$

$$\int_0^\infty A(\lambda)\cos\lambda x_1\, d\lambda = 0 \quad (a < x_1 < \infty)$$

Here for the equal roots

$$A(\lambda) = \lambda^{-1}A_1(\lambda), \quad g(\lambda) = -f(\mu_1) + \mu_1/k\operatorname{sh}^2\mu_1 \tag{3.6}$$

$$k = \frac{(l_1 - l_2)d_2}{l_1 d_1}, \quad f(\mu) = \frac{\exp(-\mu)}{\operatorname{sh}\mu}, \quad \mu_i = \lambda h_i, \quad i = 1, 2$$

For unequal roots

$$A(\lambda) = \lambda^{-1}A_2(\lambda), \ g(\lambda) = k^{-1}[k_2 f(\mu_1) - k_1 f(\mu_2)] \tag{3.7}$$

$$k_1 = l_1 n_2^{-1/2}, \ k_2 = l_2 n_1^{-1/2}, \ k = k_1 - k_2$$

The symmetric buckling mode. On the basis of the first two conditions in (1.3) we obtain a system of dual integral Eqs. (3.5), where in the case of equal roots

$$A(\lambda) = A_2(\lambda), \ g(\lambda) = -f(\mu_1) - \mu_1/(k\operatorname{sh}^2\mu_1) \tag{3.8}$$

For unequal roots

$$A(\lambda) = A_1(\lambda), \ g(\lambda) = k^{-1}[k_2 f(\mu_2) - k_1 f(\mu_1)] \tag{3.9}$$

4. We will seek the solution of system (3.5) in the form /4/

$$A(\lambda) = \lambda^{-2}\int_0^a \omega(t)(\cos\lambda t - \cos\lambda a)\, dt \tag{4.1}$$

where $\omega(t)$ is an unknown function. It can be shown, by taking account of the value of the discontinuous integral /5/

$$\int_0^\infty \frac{\sin px \cos qx}{x}\, dx = \begin{cases} \pi/2, & 0 \leqslant q < p \\ \pi/4, & q = p \\ 0, & q > p \geqslant 0 \end{cases}$$

and using integration by parts, that the second equation in (3.5) is satisfied. The remaining dual equation when taking account of the value of the integral /5/

$$\int_0^\infty \frac{(\cos\lambda t - \cos\lambda a)\cos\lambda x_1}{\lambda}\, d\lambda = \frac{1}{2}\ln\left|\frac{a^2 - x_1^2}{t^2 - x_1^2}\right|$$

enables an integral equation to be obtained with a logarithmic singularity

$$\int_0^a \omega(t)\ln\left|\frac{a^2 - x_1^2}{t^2 - x_1^2}\right| dt - \int_0^a \omega(t) K(x_1, t)\, dt = 0 \tag{4.2}$$

$$K(x_1, t) = 2\int_0^\infty \lambda^{-1} g(\lambda)(\cos\lambda t - \cos\lambda a)\cos\lambda x_1\, d\lambda \tag{4.3}$$

where $g(\lambda)$ is determined on the basis of (3.6)-(3.9)

Using the unknown integrals /5/, the properties of the Gamma function and its logarithmic derivative /6/, we obtain the integrals needed later to write the kernel of (4.3) in explicit form

$$\int_0^\infty \frac{\sin\lambda x \sin\lambda t}{\operatorname{sh}^2 \lambda h_1}\, d\lambda = \frac{1}{4h_1^2}\left\{-(x - t)\pi \operatorname{cth}\frac{(x - t)\pi}{2h_1}\right\} \tag{4.4}$$

$$\int_0^\infty \frac{\exp(-q\lambda)\sin\lambda x \sin\lambda t}{\lambda \operatorname{sh} c\lambda}\, d\lambda = \left\{\operatorname{Re}\ln\Gamma\left(\frac{q + c + i(t - x)}{2c}\right)\right\}$$

$$(\{f(x)\} = f(x) - f(-x))$$

Reducing relationships (4.2) and (4.3) to dimensionless form and taking account of (4.4), (3.6)-(3.9), we obtain the following eigenvalue problem in the reduction parameter λ_1 (see (1.1)):

$$\int_0^1 f(\eta)\ln\left|\frac{1 - \xi^2}{\eta^2 - \xi^2}\right| d\eta - \int_0^1 f(\eta) M(\xi, \eta)\, d\eta = 0, \tag{4.5}$$

$$0 \leqslant \xi < 1, \quad 0 \leqslant \eta \leqslant 1$$

$$M(\xi, \eta) = R(\eta + \xi) + R(\eta - \xi) - R(1 + \xi) - R(1 - \xi)$$

Here for equal roots

$$R(\zeta) = (-1)^\alpha \frac{\zeta\pi}{2\beta_1 k}\operatorname{cth}\frac{\zeta\pi}{2\beta_1} - 2\operatorname{Re}\ln\Gamma\left(1 + i\frac{\zeta}{2\beta_1}\right) \tag{4.6}$$

For unequal roots

$$R(\zeta) = \frac{2}{k}\left[k_2 \operatorname{Re}\ln\Gamma\left(1 + i\frac{\zeta}{2\beta_\alpha}\right) - k_1 \operatorname{Re}\ln\Gamma\left(1 + i\frac{\zeta}{2\beta_\gamma}\right)\right] \tag{4.7}$$

where $\beta_j = n_j^{-1/2}\beta$, $j = 1, 2$; $\beta = ha^{-1}$; $\alpha = 2$, $\gamma = 1$ for the symmetric buckling mode, and $\alpha = 1$, $\gamma = 2$ for the bending mode.

The kernel $M(\xi, \eta)$ of the integral Eq.(4.5) obtained is continuous everywhere, as follows from (4.6) and (4.7), except at the point λ_1^* corresponding to the surface instability of the half-space and defined by the equation $k = 0$. From reasoning of a physical nature the critical values of λ_1 in the problem under consideration should be greater than the critical values of λ_1 for a plane with a crack that equal λ_1^* /2/, i.e., they should be sought in the domain $\lambda_1^* < \lambda_1 < 1$, where the function $M(\xi, \eta)$ is continuous.

5. It is convenient to carry out a numerical investigation of the eigenvalue problems (4.5)-(4.7) obtained for the parameter λ_1 by the Bubnov-Galerkin method. The numerical integration is performed by Gauss quadrature formulas and quadrature formulas for the integration of functions with a logarithmic singularity /7/. The function $\ln\Gamma(z)$ in the kernel $M(\xi, \eta)$ is calculated effectively by using asymptotic expansions /6/ or rational approximations /8/

with recursion relations. The system of power functions $1, x, x^2, \ldots$ was selected as the coordinate system in the examples presented below.

Examples. 1°. A material with a Bartenev-Khazanovich potential (an incompressible body, equal roots). In the case of the subcritical state (1.1) we have for the potential under consideration (μ is a material constant)

$$\begin{aligned}
&\varkappa_{1111} = 2\mu\lambda_1^{-2}\lambda_2,\ \varkappa_{1122} = 0,\ \varkappa_{1212} = 2\mu\lambda_1^{-1}\lambda_2(\lambda_1+\lambda_2)^{-1}\\
&\varkappa_{2112} = \varkappa_{1221} = 2\mu(\lambda_1+\lambda_2)^{-1},\ \varkappa_{2222} = 2\mu\lambda_2^{-1},\ \lambda_1\lambda_2\lambda_3 = 1,\ \lambda_3 = 1\\
&n_1 = n_2 = \lambda_1^2,\ m_1 = 1,\ m_2 = 1,\ c_{44} = 2\mu\lambda_1^{-1}(1+\lambda_1^2)^{-1},\ \beta_1 = \beta\lambda_1^{-1}\\
&l_1 = 1,\ l_2 = 2^{-1}(\lambda_1^{-2}-1),\ k = (3\lambda_1^2-1)(\lambda_1^2+1)^{-1},\ \lambda_1^* = 0.577
\end{aligned} \tag{5.1}$$

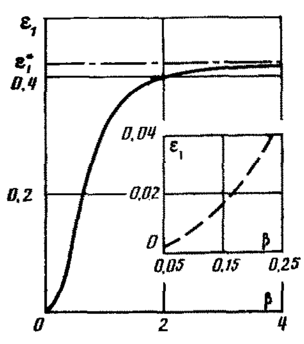
The dependence of the relative critical reduction $\varepsilon_1 = (1-\lambda_1)$ on the dimensionless half-distance between the cracks $\beta = ha^{-1}$ for the bending buckling mode is represented by the solid line in the figure (for the symmetric mode the critical values of ε_1 turned out to be greater than the values $\varepsilon_1^* = (1-\lambda_1^*)$, corresponding to the surface instability of the half-plane). The nature of the convergence as a function of the number N of coordinate functions utilized is illustrated for $\beta = 1/_8$ by the following data: $\varepsilon_1 = 0.008$ for $N = 1$, $\varepsilon_1 = 0.010$ for $N = 2$ and $\varepsilon_1 = 0.012$ for $N = 3, 4, 5$ (the critical values ε_1 are presented to the third decimal place).

2°. A material with a Treloar potential (an incompressible body, unequal roots). We have for the subcritical state (1.1) (C_{10} is a material constant)

$$\begin{aligned}
&\varkappa_{1111} = 2C_{10}(\lambda_1^{-2}\lambda_2^2+1),\ \varkappa_{1122} = 0,\ \varkappa_{1212} = 2C_{10}\lambda_1^{-1}\lambda_2\\
&\varkappa_{1221} = \varkappa_{2112} = 2C_{10},\ \varkappa_{2222} = 4C_{10};\ c_{44} = 2C_{10}\lambda_1^{-1}\lambda_2\\
&n_1 = 1,\ n_2 = \lambda_1^2\lambda_2^{-2},\ m_1 = \lambda_1^{-1}\lambda_2,\ m_2 = \lambda_1\lambda_2^{-1}\\
&l_1 = 2\lambda_1\lambda_2(\lambda_1^2+\lambda_2^2)^{-1},\ l_2 = 2^{-1}\lambda_1^{-1}\lambda_2(1+\lambda_1^2\lambda_2^{-2})\\
&\beta_1 = \beta,\ \beta_2 = \beta\lambda_1^{-2},\ k_1 = 2\lambda_2^2(\lambda_1^2+\lambda_2^2)^{-1},\ k_2 = 2^{-1}\lambda_1^{-1}\lambda_2(1+\lambda_1^2\lambda_2^{-2}),\\
&k = k_1 - k_2,\ \lambda_1\lambda_2\lambda_3 = 1,\ \lambda_3 = 1,\ \lambda_1^* = 0.544
\end{aligned} \tag{5.2}$$

For the material mentioned, the dependence of the relative critical reduction ε_1 on the dimensionless half-distance between cracks β is represented by the dashed line in the figure (the results are presented for the bending mode).

Discussion of the results. As the results of the numerical analysis show, the method used for the investigation is quite effective (two-three coordinate functions are adequate for calculating the critical value of the relative reduction to the third decimal place $\varepsilon_1 = (1-\lambda_1)$. For small values of the dimensionless half-distance between cracks $\beta = ha^{-1}$ the mutual influence of the cracks results in the critical values of ε_1 being considerably less (by two orders of magnitude) than the values corresponding to one crack in an infinite material and equal ε_1^* /2/ (see the figure).



As $\beta \to \infty$ the critical values of the relative reduction tend asymptotically to the value $\varepsilon_1 = \varepsilon_1^*$ both for the bending and the symmetric buckling modes (as $\beta \to \infty$ the cracks do not interact, i.e., this is the case of an isolated crack in an infinite material).

In conformity with the fracture criterion assumed the critical values obtained for the relative reduction ε_1 characterize the beginning of the fracture of a material weakened by a periodic system of parallel cracks under compression along the latter. In the case under consideration when the bending buckling mode is realized, the critical values of ε_1 obtained will here obviously also characterize the subsequent total fracture of the material in the whole domain occupied by the cracks since a phenomenon occurs in this case that is analogous to the appearance of a plastic hinge over the whole material thickness in the beam bending case.

A completely different situation is obtained in the case of near-surface cracks /3/, when local buckling results just in local fracture of the interlayer between the crack and the free surface and the question of fracture of the whole material requires further study. An analogous phenomenon also holds for a finite number of cracks when local buckling also results in just local fracture; the questions of local fracture as it applies to the case of two cracks are investigated in /9/.

Note that the method of investigation developed above for elastic materials weakened by a periodic system of cracks can also be extended to more complex models of materials /2, 10, 11/.

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$$\int_0^1 x^\alpha \ln\frac{e}{x} f(x)\,dx, \quad \int_0^1 x^\beta \ln\frac{e}{x}\ln\frac{e}{1-x} f(x)\,dx$$

$$\int_0^1 \ln\frac{1}{x} f(x)\,dx, \quad \int_0^\infty x^\beta e^{-x} \ln\left(1+\frac{1}{x}\right) f(x)\,dx$$

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FRACTURE CRITERIA FOR MATERIALS WITH DEFECTS*

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Classical theories of the strength of materials start from concepts of the existence and uniqueness of fracture surfaces in the space of indpendent loading parameters: on approaching a certain point of this surface from within along any arbitrary loading path, the instant of fracture is fixed by the very same combination of loading parameters. Such are all the strength criteria appied in the strength of materials (in stress space), for instance, Galileo, Poncelet, Coulomb, Tresca, Saint-Venant, Moore, Mises, etc. /1-16/. This concept turned out to be valid even from the viewpoint of fracture mechanics in the case of active loading paths /17/.

Analysis of these concept in the case of two (and more) independent loading parameters and for any loading paths is of interest from the viewpoint of modern fracture mechanics according to which the fracture of real materials is explained by the development of cracks in them from certain initial defects. The most widespread kinds of initial defects here are obviously pores and cracks. Representative of crack and pore materials are concrete, ceramics, composites, mountain rocks and other geomaterials for which the representation of a fracture surface is used extensively at present to describe their strength.

We consider below two problems of fracture mechanics with two independent loading

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